Graph Theory

1. Applications:
   - World Wide Web
   - Scheduling
   - Chip Design
   - Network Analysis
   - Flow Charts

2. Definitions:
   1. A graph $G=(V,E)$ consists of a set of vertices (nodes), denoted by V, and a set of edges, denoted by E.

   2. $n=|V|$. I.e. n is the number of nodes.

   3. $m=|E|$. I.e. m is the number of edges.

   4. In an undirected graph, each edge is a set of 2 vertices, $\{u,v\}$. This makes (u,v) and (v,u) the same. Furthermore, self-loops are not allowed.
      Note: When it's clear from context, we will use (u,v) for $\{u,v\}$.

   5. In a directed graph, each edge is an ordered pair of nodes. Therefore, (u,v) is different from (v,u). Furthermore, self-loops are allowed. This means that (u,u) is allowed.

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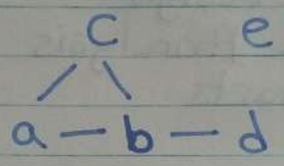
6. Two vertices are adjacent iff there is an edge between them.

E.g. Consider the graph below.



We can store which nodes are adjacent in 2 ways.

1. Adjacency Matrix

	a	b	c	d	e
a		✓	✓		
b	✓		✓	✓	
c	✓	✓			
d		✓			
e					

- An adjacency matrix is a 2-D array.

- Space: $\Theta(n^2)$
- who are adjacent to v: $\Theta(n)$ t
- are v and w adjacent: $\Theta(1)$ t
- Convenient for some other operations and queries.

2. Adjacency Lists

	Is adjacent to
a	b, c
b	a, c, d
c	a, b
d	b
e	

- With adjacency lists, we store the vertices in a 1-D array or dictionary. At entry A[i], we store the neighbours of v_i.

- If the graph is directed, we store only the out-neighbours.

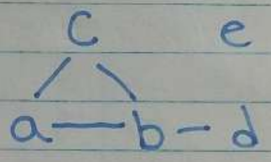
- Space: $\Theta(m+n)$
- who are adj to v:
  $\Theta(\deg(v))$ time
  I.e. Length of adj list
- are v and w adj:
  $\Theta(\deg(v))$ time if a list

- Optimal for graph searches.

7. **Traversal**: Visit each vertex of a graph.

8. **Path**: A sequence of edges which connect a sequence of distinct vertices.
I.e. You can't go through a vertex twice.

E.g. Consider the graph below.



<d> is a path of length 0.
<d, b, c> is a path of len 2.
<d, a, b> is not a path.

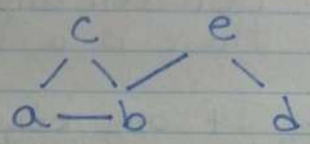
9. v is **reachable** from u iff there is a path from u to v.

10. A **simple cycle** is a non-empty sequence of vertices in which:

1. Consecutive vertices are adjacent
2. First Vertex = Last Vertex
3. Vertices are distinct, except for the first and last
4. Edges used are distinct

Note: $\langle v \rangle$ is NOT a cycle.

E.g. Consider the graph below.



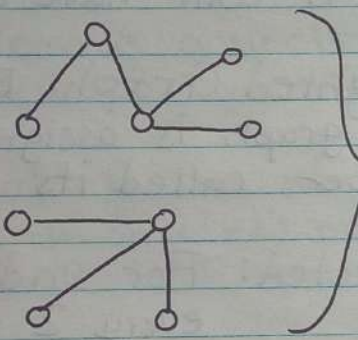
1. $\langle b, c, a, b \rangle$ is a simple cycle of length 3.

2. $\langle b, c, a, b, d, e, b \rangle$ is not a simple cycle.

3. $\langle b, d, b \rangle$ is not a cycle because it uses $\{b, d\}$ twice.

11. A tree is a graph that is connected but has no cycles.

E.g.



Trees

A forest is a collection of trees.

Note: Acyclic means that there are no cycles.

Trees have the following properties:

1. Between any 2 vertices, there is a unique path.

2. A tree is connected by default, but if an edge is removed, it becomes disconnected.

3. # edges = # vertices -1
   I.e. $m = n - 1$

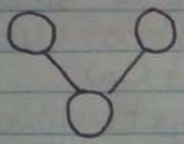
4. Acyclic by default, but if a new edge is added, then it will have a cycle.

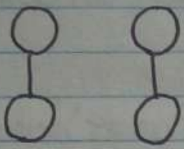
12. Weighted Graph: Each edge in the graph is assigned a real number, called its weight.

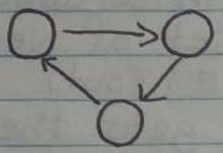
13. Connected: For undirected graphs, every 2 vertices have a path between them.

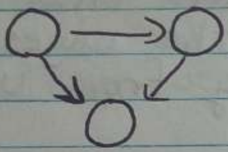
14. Strongly Connected: For directed graphs, for any 2 vertices, u, v, there is a directed path from u to v.

E.g.

1.  Connected

2.  Not Connected

3.  Strongly Connected

4.  Not Strongly Connected

3. Operations:

1. Add / Remove a vertex/edge.

2. Edge Query: Given 2 vertices, u, v, find out if the edge (u,v) (if the graph is directed) or the edge $\{u,v\}$ is in E.

3. Neighbourhood: Given a vertex u in an undirected graph, get the set of vertices $\{v \mid \{u,v\} \in E\}$.

4. In-neighbourhood: Given a vertex u in a directed graph, get the set of vertices $\{v \mid (v,u) \in E\}$ (in).
   I.e. This gets the set of vertices whose edges lead to u.

5. Out-neighbourhood: Given a vertex u in a directed graph, get the set of vertices $\{v \mid (u,v) \in E\}$ (out).
   I.e. This gets the set of vertices that can be reached by the edges that lead away from u.

6. Degree: Computes the size of the neighbourhood.

7. In-Degree: Computes the size of the in-neighbourhood.

8. Out-Degree: Computes the size of the out-neighbourhood.

4. Breadth-First Search (BFS):

1. Algorithim:

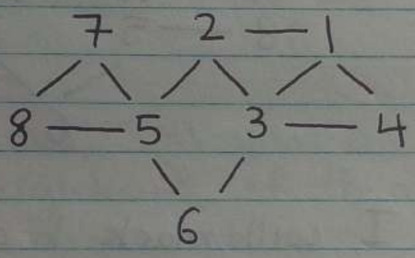
   1. Start at v. Visit v and mark as visited.

   2. Visit every unmarked neighbour of v and mark each neighbour as visited.

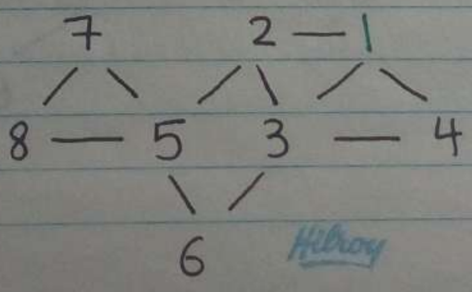
   3. Mark v finished.

   4. Recurse on each vertex marked as visited in the order they were visited.

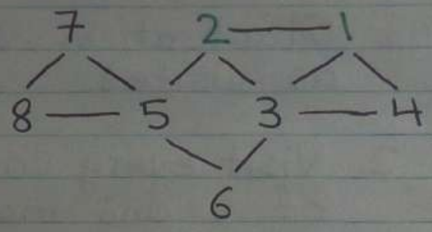
E.g. Consider the graph below.

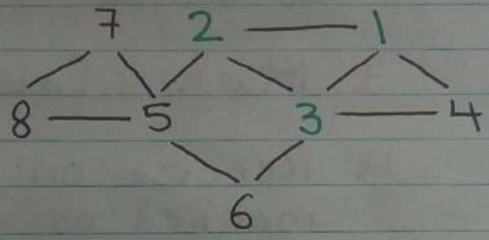


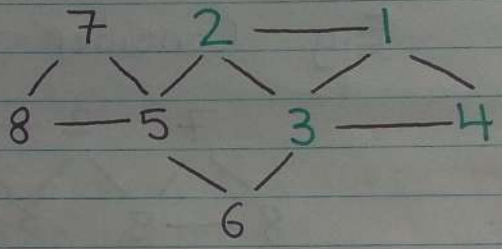
1. Start at 1. I'll mark a node as visited by writing it in green.



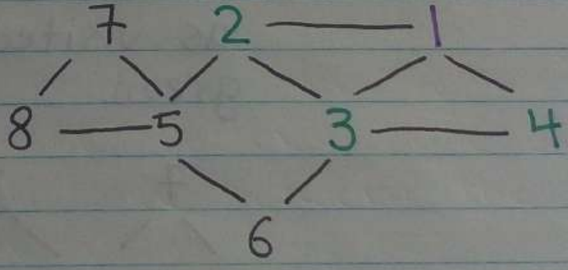
2. I will visit all the unmarked neighbours of 1 in the following order: 2, 3, 4.



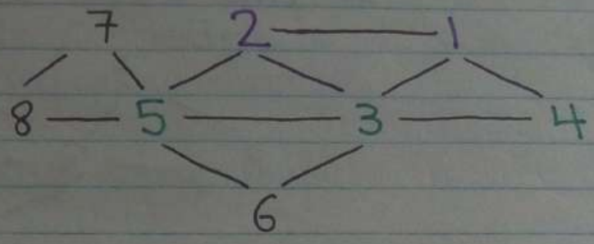




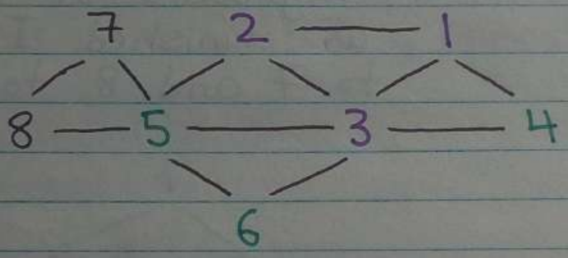
3. I will mark 1 as finished by writing it in purple.



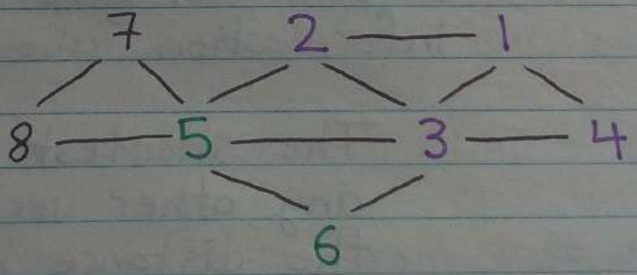
4. Since I visited 2 first, I will visit every unmarked neighbour of 2 and then mark 2 as finished.



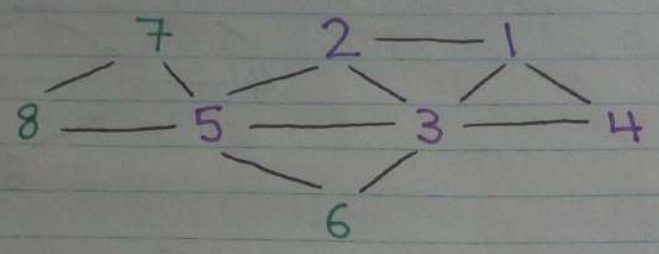
5. Next, I will visit all the unmarked neighbours of 3 and then mark 3 as finished.



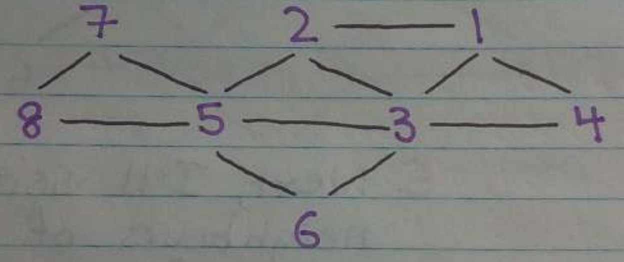
6. Next, I'll visit all the unmarked neighbours of 4 and then mark 4 as finished.



7. I'll visit all the unmarked neighbours of 5 and then I'll mark 5 as finished.
I'll visit the unmarked neighbours in this order: 7, 8.



8. I'll visit all the unmarked neighbours of 6, and mark it as finished. I'll do the same to 7 and 8, too.



A BFS can give the following information about a graph.

1. The shortest path from v to any other vertex u. We denote the distance between the nodes as d(v).

2. Whether the graph is connected.

3. The number of connected components.

A BFS constructs a tree that visits every node connected to v. We call this a spanning tree.

2. Implementing BFS:

We can use a queue to implement a BFS given an adjacency list representation of a graph.

A queue is FIFO (First in, First out) and has the following operations:

1. Enqueue(Q, v)
2. Dequeue(Q)
3. Isempty(Q)

Furthermore, we will need to store the following information for each v:

1. The current node, u, and it's state (visited, not visited, finished)

2. The predecessor, p[u]

3. The distance from u to v.

4. The order of discovery

3. Complexity:

- Since each node is enqueued at most once, the adjacency list of each node is examined at most once. Therefore, the total running time of BFS is $O(m+n)$ or linear in the size of the adjacency list.

Note: Each node is enqueued when it is not visited, at which point it is marked visited.

Note:
- BFS will only visit the nodes that are reachable from V.
- If the graph is connected (In the undirected case) or strongly-connected (In the directed case), then this will be all vertices.
  - If not, then we may have to call BFS multiple times in order to see the whole graph.

5. Depth First Search (DFS):

1. Algorithim:

- All vertices and edges start out unmarked.
- Start at vertex v and go as far as possible away from v visiting vertices.
- If the current vertex has not been visited, mark it as visited and the edge that is traversed as a DFS edge.
- If the current vertex has been visited, mark the traversed edge as a back-up edge, back up to the previous vertex.
- When the current vertex has only visited neighbours left, mark it as finished.
- Backtrack to the first vertex that is not finished.
- Continue

Just like BFS, DFS constructs a spanning-tree and gives connected component information.

However, DFS does not find the shortest distance between v and all other vertices.

2. Implementing A DFS:

- We can use a stack (LIFO) to store the edges with the usual operations:

  1. push((u,v))
  2. pop()
  3. is_empty()

- Furthermore, we need to store these data for each vertex in order to easily determine whether an edge is a back-edge or a DFS-edge:

  1. d[v] will indicate the discovery time.
  2. f[v] will indicate the finish time.

3. Complexity of DFS:

- A DFS visits the neighbours of a node exactly once. Therefore, the adjacency list of each vertex is visited at most once. So, the total running time is $\Theta(n+m)$.
  I.e. Linear in the size of the adjacency list.

- Note: The DFS edges form a tree called the DFS tree. However, the DFS tree is NOT unique for a given graph G, starting at s.

4. DFS Edges:

- We can specify edges (u,v) in a DFS-tree according to how they are traversed during the search.

- If v is visited for the first time, then (u,v) is a tree-edge in a DFS tree.

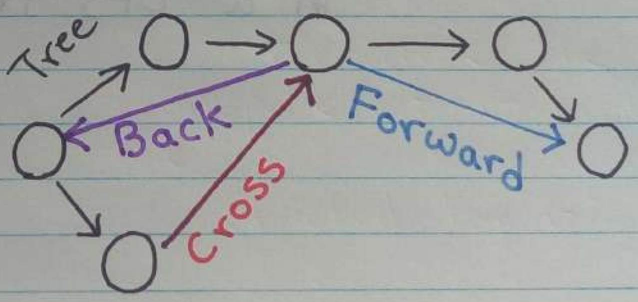
- If v has already been visited, then (u,v) is a:

1. back-edge: An edge from a vertex u to an ancestor v in the DFS tree.

2. forward-edge: An edge from a vertex u to a descendent v in the DFS tree.
   Note: This only applies to directed graphs.

3. cross-edge: All the other edges that are not part of the DFS tree.
   I.e. v is neither an ancestor nor a descendent of u in the DFS tree.
   Note: This only applies to directed graphs.

- E.g.



- We can use d[v] and f[v] to distinguish between the edges.

- There is a cycle in graph G iff there are any back-edges when DFS is run.

- We can detect a back-edge in a DFS' if the vertex we are visiting has been visited but not finished.

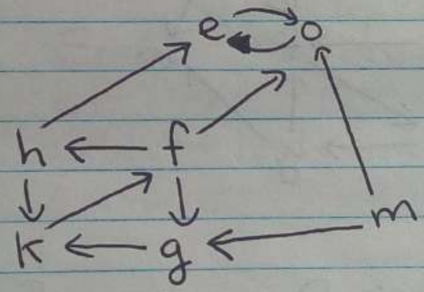
6. Strongly Connected Components (SCC):

1. Definition:

- SCC: Is the maximal subset of vertices from each other in a directed graph.

- Example:

Consider the graph below.



The scc's are:

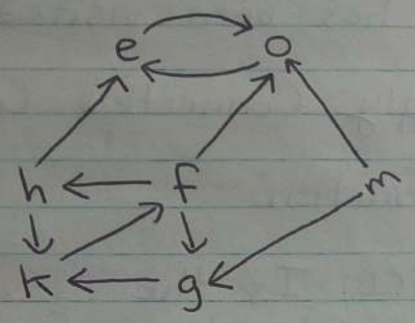
1. {e, o}
2. {m}
3. {h, f, k, g}

2. Transpose of G:

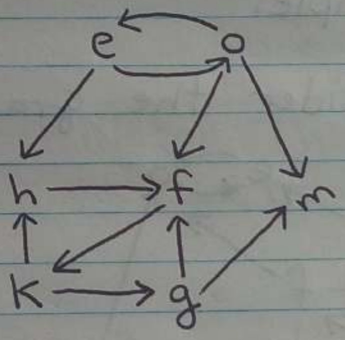
- The transpose of G, denoted by G^T, is a graph with the same vertices as G, but the edges are reversed.

- E.g.

G:



G^T:



- Note: Do not confuse the transpose of G with the complement of G, denoted by G^c.

The complement of G is all possible edges minus all the existing edges.

- Note: G^T has the same SCC as G.

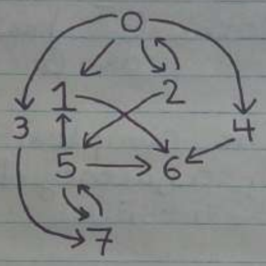
- The complexity of computing an adjacency list of G^T is $O(|V| + |E|)$.

7. Kosaraju's SCC Algorithim:

1. Overview:
   - DFS on G. Visit all the vertices, note finish times and accumulate vertices in reverse finishing order.
   - Compute the adjacency lists of G^T.
   - DFS on G^T, using the above order to pick start/restart vertices.
   - Each tree found has the vertices of one SCC. In total, this takes $O(|V| + |E|)$ time.

2. Example:

Consider the graph below.



1. Start at 0 and go to 1.
I will store the visited nodes in a list and the finished vertices in a separate list.

visited: 0, 1
finished:

2. From 1, I will go to 6.

visited: 0, 1, 6
finished:

3. Since there is nowhere to go from 6, I will mark it as finished and backtrack to 1.

visited: 0, 1, 6
finished: 6

4. Since there is nowhere to go from 1, I will mark it as finished and backtrack to 0.

visited: 0, 1, 6
finished: 6, 1

5. From 0, I will visit 2.

visited: 0, 1, 6, 2
finished: 6, 1

6. From 2, I will go to 5.
Note: I cannot go to 0 because I have already visited it.

visited: 0, 1, 6, 2, 5
finished: 6, 1

7. From 5, I will go to 7.
Note: I can't go to 1 or 6 as I have visited them already.

visited: 0, 1, 6, 2, 5, 7
finished: 6, 1

8. There is nowhere to go from 7, so I will mark it as finished and backtrack to 5.

visited: 0, 1, 6, 2, 5, 7
finished: 6, 1, 7

4

9. There is nowhere to go from 5, so I will mark it as finished and back-track to 2.

visited: 0, 1, 6, 2, 5, 7
finished: 6, 1, 7, 5

10. There is nowhere to go from 2, so I will mark it as finished and backtrack to 0.

visited: 0, 1, 6, 2, 5, 7
finished: 6, 1, 7, 5, 2

11. From 0, I will visit 3.

visited: 0, 1, 6, 2, 5, 7, 3
finished: 6, 1, 7, 5, 2

12. There is nowhere to go from 3, as I have already visited 7, so I will mark it as finished and backtrack to 0.

visited: 0, 1, 6, 2, 5, 7, 3
finished: 6, 1, 7, 5, 2, 3

13. From 0, I will visit 4.

visited: 0, 1, 6, 2, 5, 7, 3, 4
finished: 6, 1, 7, 5, 2, 3

4

14. There is nowhere to go from 4, so I will mark it as finished and backtrack to 0.

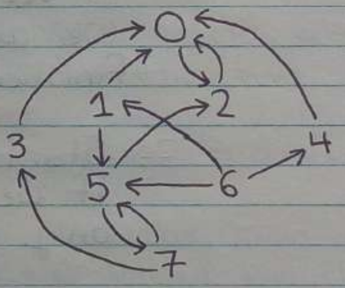
visited: 0, 1, 6, 2, 5, 7, 3, 4
finished: 6, 1, 7, 5, 2, 3, 4

15. There is nowhere to go from 0, so I will mark it as finished.
Note: The first node visited is also the last node visited.

visited: 0, 1, 6, 2, 5, 7, 3, 4
finished: 6, 1, 7, 5, 2, 3, 4, 0

16. Now, we find G^T, reverse the finished list and DFS G^T based on the ordering of the new finished list.

G^T:



finished: 0, 4, 3, 2, 5, 7, 1, 6

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17. Starting at 0, if we do a DFS, the only vertex we can reach is 2.

$\therefore$ SCC #1 = $\{0, 2\}$
Furthermore, we can remove 0 and 2 from the finished list.

18. Starting at 4, we see that there is nowhere to go.

$\therefore$ SCC # 2 = $\{4\}$
We can remove 4 from the finished list.

19. Starting at 3, we see that there is nowhere to go.

$\therefore$ SCC # 3 = $\{3\}$
We can remove 3 from the finished list.

20. Starting at 5, we see that if we do a DFS, we can only go to 7.

$\therefore$ SCC # 4 = $\{5, 7\}$
We can remove 5 and 7 from the finished list.

21. Starting at 1, we see that there is nowhere to go.
$\therefore$ SCC # 5 = $\{1\}$
We can remove 1 from the finished list.

22. Starting at 6, we see that there is nowhere to go.

$\therefore$ SCC #6 = $\{6\}$

$\therefore$ In total, SCC = $\{0,2\}$, $\{4\}$, $\{3\}$, $\{5,7\}$, $\{1\}$, $\{6\}$

3. Proof of kosaraju's Algorithim:

1. Notation:

- We denoted $f(v)$ as the time at which vertex v is finished.
- $f(u) < f(v)$ means u is finished before v.
- Let C be an SCC. We define $f(C)$ to be the time at which the last node in C finishes. Formally, $f(C) = \max_{v \in C} f(v)$

2. Lemma: If s is the first node in SCC C visited by DFS, then $f(C) = f(s)$.

Proof:
Since s is the first node in C visited by DFS, all vertices in C are not finished. Furthermore, since C is a SCC,

every vertex in C is reachable from s. That means there is a path from s to every vertex in C. Thus, every node will be finished when DFS returns. Since the last step of the DFS is to finish s, this means that s is finished only after all other vertices are finished. Therefore, $f(s) > f(v)$ for any $v \in C$. By the definition of $f(C) = \max_{v \in C} f(v)$, $f(C) = f(s)$.

3. Thm: Suppose we run DFS starting at each node in G. Let C_1 and C_2 be SCCs in G. If (u, v) is an edge in G where $u \in C_1$ and $v \in C_2$, then $f(C_2) < f(C_1)$.

Proof:
- Let x_1 and x_2 be the first vertices DFS visits in C_1 and C_2, respectively.
- By our lemma, $f(C_1) = f(x_1)$ and $f(C_2) = f(x_2)$. Therefore, we will show $f(x_2) < f(x_1)$.
- Note: x_2 is reachable from x_1, because there is a path from x_1 to u in C_1, across (u, v) and a path from v to x_2 in C_2.

However, x_1 is not reachable from x_2, since then x_1 and x_2 would be strongly connected, contradicting that they belong in different SCCs.

- We have 2 cases:

1. $DFS(x_2)$ is called before $DFS(x_1)$:

   - Since x_1 is not reachable from x_2, x_2 will finish before x_1.

     $\therefore f(x_2) < f(x_1)$, as wanted

2. $DFS(x_1)$ is called before $DFS(x_2)$:

   - When $DFS(x_1)$ is called, all nodes in C_1 and C_2 have not been visited, so there is a DFS path from x_1 to x_2.

   - When $DFS(x_1)$ returns, x_2 will be finished.

   - Since x_1 will be finished just before $DFS(x_1)$ returns, this means that x_1 finished after x_2, so $f(x_2) < f(x_1)$.

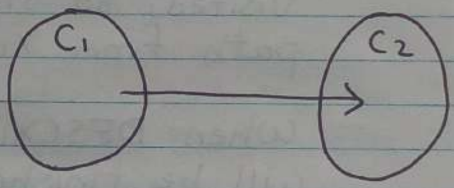
4. Corollary: Let C_1 and C_2 be distinct SCCs in $G = (V, E)$. Suppose there is an edge (u, v) in E^T where $u \in C_1$ and $v \in C_2$. Then $f(C_1) < f(C_2)$

5. Corollary: Let C_1 and C_2 be two distinct SCCs in $G = (V, E)$. If $f(C_1) > f(C_2)$, then there cannot be an edge from C_1 to C_2 in G^T.

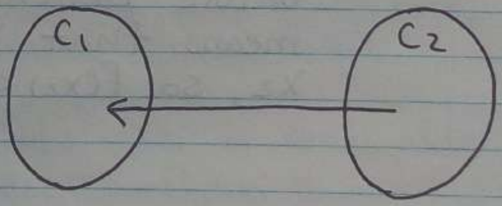
Consider this:

- Since we know that $f(C_1) > f(C_2)$, then there is an edge from C_1 to C_2. However, in G^T, that edge is reversed, so there is no longer an edge from C_1 to C_2.

G:



G^T:



This means that if we start the DFS on G^T at C_1, because there is no edge from C_1 to C_2, the DFS will only visit the vertices from C_1 and it will return a DFS tree that contains only vertices from C_1. Then, when you do DFS on C_2, even though there is an edge from C_2 to C_1, DFS will only visit the vertices in C_2 because we already finished C_1. We continue for all remaining SCCs.

Proof:

- Edge $(u,v) \in E^T$ implies $(v,u) \in E$.

- Since SCCs of G and G^T are the same, $f(C_2) > f(C_1)$.

- This completes the proof.